5.3 Binomial Distributions

Parvin Das is a quality-control engineer. One of his responsibilities is to monitor the defect rate of a production line. Probability distributions based on the results of the Binomial Theorem can be used as mathematical models to do this.

Binomial Experiments

Each time a quality-control inspector chooses an item to test, there are two possible outcomes for the test. The item either passes the test and is used, or it fails and does not get used. Each of these outcomes has a probability associated with it.

To guarantee consistency, the quality-control process is based on certain assumptions. These assumptions form the formal definition of a binomial experiment.

Binomial Experiment

A binomial experiment is any experiment that has the following properties:

1. There are $n$ identical trials. Together, these form a binomial experiment.
2. The purpose of the experiment is to determine the number of successes that occurs during the $n$ trials.
3. There are two possible outcomes for each trial. These are usually termed success and failure. The probability of a success is usually denoted $p$ and the probability of a failure is $q$ or $1 - p$.
4. The probability of the outcomes remains the same from trial to trial. The values of $p$ and $1 - p$ do not change from one trial to the next because the value of $p$ is the same for each trial.
5. The trials are independent of one another.

Repeated trials, which are independent and have two possible outcomes (success and failure) are called Bernoulli trials. They are named after Daniel Bernoulli, a member of a remarkable family of mathematicians and scientists. Daniel’s grandfather Nicolaus Sr., his uncles Jacob I and Nicolaus I, his father John I, his brothers Nicolaus III and John II, and his nephews John III and Jacob II all made significant contributions to mathematics and science. Each member of the Bernoulli family was known for his keen intellect and fiery temper. After winning the French Academy of Science Prize, which his father had unsuccessfully tried to win, Daniel was thrown out of the house!
INVESTIGATION: THE DISTRIBUTION OF THE NUMBER OF HEADS IN FOUR COIN TOSSES

Suppose you were to simulate sampling a production line known to have a 50% defect rate. You do this by tossing coins. Calculate the probability distribution for selecting four items from this production line.

The tossing of a coin is an example of a Bernoulli trial. An experiment in which each trial requires the toss of a coin several times and the recording of the number of heads is an example of a binomial experiment.

Purpose
To investigate the results of a binomial experiment.

Procedure
A. Work with a partner and record the results of tossing four coins simultaneously. Alternatively, you could create a simulation of the coin toss using a spreadsheet, a TI-83 Plus calculator, or Fathom™.
B. Record the number of heads. A head will represent a defective item.
C. Repeat until you have completed 100 repetitions of the experiment.
D. Create a relative frequency distribution for the number of heads (defects) in the toss of four coins.
E. Either repeat Steps A through C, or combine results with other students to increase the number of trials.
F. Create a new relative frequency distribution using the larger data set.

Discussion Questions
1. How did increasing the number of repetitions of the experiment affect the shape of the distribution?
2. Why is tossing a single coin once a trial?
3. Why is each trial a Bernoulli trial?
4. Why is the number of heads obtained in each repetition a discrete random variable?

THE PROBABILITY DISTRIBUTION FOR A DICE-ROLL SIMULATION

In a binomial experiment, the number of successes in n repeated Bernoulli trials is a discrete random variable, usually represented by the letter X. X is termed a binomial random variable and its probability distribution is called a binomial distribution.

Rolling a Die
Suppose a cereal company puts one of six possible prizes in each cereal box. You can use a simulation involving rolling a six-sided die to determine the probability distribution for getting a particular prize in four purchased boxes of cereal.
Define a success as the appearance of a 1. What is the probability for one, two, three, four, or zero 1s showing?

The table in the margin shows the results of 30 repetitions of a simulation of this experiment using a spreadsheet. The frequency and experimental probability distribution for all 100 repetitions of the simulation appear below.

### Analysis of the Simulation as a Counting Problem

Verify that the die-roll simulation is a binomial experiment by checking that it has all the required properties. Then, conduct the experiment yourself or do a simulation to see if your experimental probability distribution is the same.

1. There were four identical trials in which a die was rolled. A trial involves rolling a die one time. Each repetition of the experiment consists of four trials. You would then count the number of trials that produces a number 1, in each of the 100 repetitions of the experiment.

2. The trials were Bernoulli trials because
   - there were only two possible outcomes—a success is a roll of 1, and a failure is a roll that is not a 1. In this case, \( p = \frac{1}{6} \) and \( 1 - p = \frac{5}{6} \);
   - the probability of success is the same for every roll;
   - the trials are independent of one another; and
   - the purpose of the experiment is to determine the number of 1s that occurs in four rolls. That is the binomial random variable.

Think about how to record the possible outcomes of the four rolls of the die. For each repetition of the experiment, record the outcome of each of the four trials in a table similar to the following example.
Example 1 Rolling 1s

(a) What is the probability that the first roll will be a 1 and all the others will be something other than a 1?

(b) Find the probability of the combined event that the roll of 1 will appear in any of the four available positions in the table.

(c) What is the probability that exactly two 1s show in the four rolls of the die?

(d) Complete the theoretical probability distribution for the number of 1s showing in four rolls.

Solution

(a) Consider the following:

\[
P(\text{Roll 1} = 1) = \frac{1}{6} \quad P(\text{Roll 2} \neq 1) = \frac{5}{6}
\]

\[
P(\text{Roll 3} \neq 1) = \frac{5}{6} \quad P(\text{Roll 4} \neq 1) = \frac{5}{6}
\]

Thus,

\[
P(\text{Roll 1} = 1) \text{ AND } \text{Roll 2} \neq 1 \text{ AND } \text{Roll 3} \neq 1 \text{ AND } \text{Roll 4} \neq 1)
\]

\[
= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}
\]

\[
= \frac{1}{6} \left(\frac{5}{6}\right)^3
\]

(b) The number of ways that 1 can be placed in one of the four entries of the table is the same as counting the number of ways one object can be selected from four available objects. This can be done in \(\binom{4}{1} = 4\) ways. Therefore, the probability that only one of the rolls will result in a 1 showing is the sum of the four individual probabilities, all of which are \(\frac{1}{6} \left(\frac{5}{6}\right)^3\). Therefore,

\[
P(\text{one 1 in four trials}) = \binom{4}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^3
\]

\[
= \frac{500}{1296} \text{ or about 0.39}
\]

(c) Suppose the two 1s appeared in Roll 1 and Roll 2. The probability of this event is

\[
P(\text{R1} = 1 \text{ AND R2} = 1 \text{ AND R3} \neq 1 \text{ AND R4} \neq 1)
\]

\[
= \left(\frac{1}{6}\right) \left(\frac{1}{6}\right) \left(\frac{5}{6}\right) \left(\frac{5}{6}\right)
\]

\[
= \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2
\]

The number of ways that exactly two of the entries can be filled in with 1s is \(\binom{4}{2} = 6\).

Therefore, \(P(\text{two 1s in four trials}) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2\) or about 0.12.
Each row of the probability distribution for this experiment can be found using reasoning similar to that on the previous page.

The probability that the four rolls of the die will show $k$ 1s is given by the formula

$$P(k \text{ 1s in four trials}) = \binom{4}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{4-k}.$$  

This formula is another way to express the probability distribution of the binomial random variable that indicates the number of 1s showing in four trials. Remember that a probability distribution can be represented by a graph, a table of values, or a formula.

**BINOMIAL PROBABILITY DISTRIBUTION**

In general, the probability of $k$ successes in $n$ trials of a binomial experiment corresponds to finding the number of ways the $k$ successes can be recorded in the $n$ available recording slots in the outcome table for the event. There are $\binom{n}{k}$ ways of selecting the locations in which to record the successes. The probability of each of these is $(p)^k(q)^{n-k} = (p)^k(1 - p)^{n-k}$. Using the Additive Principle for Probabilities, the probability of the event corresponding to $k$ successes is the sum of all the individual probabilities for the outcomes that make up the event.

### Binomial Probability Distribution

Consider a binomial experiment in which there are $n$ Bernoulli trials, each with a probability of success of $p$. The probability of $k$ successes in the $n$ trials is given by

$$P(X = k) = \binom{n}{k} (p)^k (1 - p)^{n-k}$$

where $X$ is the discrete random variable corresponding to the number of successes.

**Example 2 Cost of Coffee**

In Section 4.1, you investigated the amount you could expect to pay for coffee if you and a friend tossed a coin to determine who would pay each day. Coffee costs $1.00 a cup. Determine the expected cost to you each week.

**Solution**

Assume that each coin toss is a Bernoulli trial. Your correct call is defined as a success for which $p = \frac{1}{2}$. The experiment consists of five trials (one for each day of the week). Consider the number of successes. The discrete random variable, $X$, represents the number of wins in five tosses. This is a binomial experiment for which the binomial probability distribution formula yields

$$P(X = k) = \binom{5}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{5-k} = \binom{5}{k} \left(\frac{1}{2}\right)^5$$
If we examine the expected cost of the coffee game, we also observe the following:

\[ E(X) = 1 \left( \frac{5}{1} \right) \left( \frac{1}{2} \right)^5 + 2 \left( \frac{5}{2} \right) \left( \frac{1}{2} \right)^5 + ... + 5 \left( \frac{5}{5} \right) \left( \frac{1}{2} \right)^5 \]

\[ = 2.5 \]

In this case, the number of trials multiplied by the probability of success results in the expected value.

**Expected Value of a Binomial Experiment**

The expected value of a binomial experiment that consists of \( n \) Bernoulli trials with a probability of success, \( p \), on each trial is

\[ E(X) = np \]

**Example 3 Probability of a Hit**

In Section 4.1, you considered a baseball player who had a batting average of 0.320. You created a simulation to predict the likelihood that he could have a game in which he has no hits in three times at bat.

(a) Determine the theoretical probability of this event.

(b) Determine the probability distribution for the number of hits per game.

(c) Calculate the batter’s expected number of hits per game.

**Solution**

Assume that each time at bat is a Bernoulli trial for which a success is a hit and for which \( p = 0.320 \). The experiment consists of three trials in which the number of hits is considered a success. For the calculation of the theoretical probability, the number of games played during the season is not relevant.
The discrete random variable, $X$, represents the number of hits in three times at bat.

(a) This is a binomial experiment for which the binomial distribution formula yields

$$P(X = 0) = \binom{3}{0}(0.320)^0(1 - 0.320)^3$$
$$= 0.314 432$$

(b) The graph of the distribution is shown below.

If $X$ is the number of hits in three times at bat, the general formula for this distribution is

$$P(X = k) = \binom{3}{k}(0.320)^k(0.680)^{3-k}$$

(c) Method 1

$$E(X) = \sum_{i=1}^{n} X_i P(X_i)$$
$$= 0P(X = 0) + 1P(X = 1) + 2P(X = 2) + 3P(X = 3)$$
$$= 0.96$$

Method 2

$$E(X) = np$$
$$= 3 \times (0.32)$$
$$= 0.96$$

**KEY IDEAS**

binomial experiment—any experiment that consists of $n$ Bernoulli trials and for which the purpose of the experiment is to determine the number of successes that occurs during the $n$ trials
Bernoulli trial—Bernoulli trials have the following properties:

- There are two possible outcomes for each trial, which are usually termed success and failure. The probability of a success is usually denoted \( p \) and the probability of a failure is \( q = 1 - p \).
- The probability of the outcomes remains the same from trial to trial. The values of \( p \) and \( 1 - p \) do not change from one trial to the next.
- The trials are independent of one another.

Binomial probability distribution—consider a binomial experiment in which there are \( n \) Bernoulli trials, each with a probability of success of \( p \). The probability of \( k \) successes in the \( n \) trials is given by

\[
P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

where \( X \) is the discrete random variable corresponding to the number of successes.

Expected value of a binomial experiment—in any binomial experiment, the expected number of successes is given by

\[
E(X) = \sum_{i=1}^{n} x_i P(X = x_i) = np
\]

where \( n \) is the number of trials and \( p \) is the probability of success on each trial.

5.3 Exercises

**A**

1. For each term, identify
   (i) the number of trials
   (ii) the probability \( p \) of a success
   (iii) the number of successes
   (a) \( \binom{10}{6} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 \)
   (b) \( \binom{7}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^4 \)

2. Evaluate the following sum.

\[
\binom{3}{0} \left(\frac{1}{4}\right)^3 + \binom{3}{1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right) + \binom{3}{2} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^2 + \binom{3}{3} \left(\frac{3}{4}\right)^3
\]

**B**

3. Suppose \( p = \frac{1}{2} \). Simplify this expression.

\[
\binom{n}{k} p^k (1 - p)^{n-k}
\]

4. Explain why this sum is equal to 1.

\[
\binom{n}{0} (p)^0 (1 - p)^{n-0} + \binom{n}{1} (p)^1 (1 - p)^{n-1} + \ldots + \binom{n}{n} (p)^n (1 - p)^{n-n}
\]
5. Look back at Example 1 in this section. Use it to answer the following questions.
   (a) The expression for the probability of the specific event that a 1 appears only in the first row of the outcome table has a probability \( \left( \frac{1}{6} \right)^1 \left( \frac{5}{6} \right)^3 \). Explain.
   (b) Why is the probability in part (a) multiplied by \( \binom{4}{1} \) to find the probability of getting a single 1?
   (c) Determine the probability that an even number shows in only the second and third of four rolls of a six-sided die.
   (d) Determine the probability for an even number showing in exactly two of the four rolls of a six-sided die.
   (e) Determine the probability distribution for the number of times an even number shows in four rolls of a six-sided die.
   (f) Suppose you were tossing a coin four times instead of rolling a die. What is the probability of getting a head on the first toss only?

6. **Communication** Explain why the problems in parts (a) and (b) can be modelled using a binomial distribution, but the problems in parts (c) and (d) cannot.
   (a) Find the probability that a customer will seek a refund because of a defective product when traditionally 10% of all customers have requested such a refund.
   (b) Find the probability that 3 defective parts will show up in a sample of 10 parts selected randomly from a manufacturing process that the plant knows has a 5% defect rate.
   (c) Find the probability that 3 defective parts will show up in a sample of 10 parts selected randomly from a manufacturing process when it is known that there are 3 defective machines out of 10.
   (d) A hockey goaltender has stopped 387 of 400 shots. Find the probability that she will stop the next 3 shots in a row.

Design simulations for Questions 7 to 11. Compare the probability distribution resulting from your simulation with the theoretical distribution in each question.

7. **Knowledge and Understanding** Mail-order marketing companies have a response rate of 15% to their advertising flyers.
   (a) Compute the probability that exactly 3 people out of a sample of 20 respond to the flyers they receive.
   (b) Find the expected number of people in a sample of 20 who will respond to the flyers.
   (c) Compute the probability that at least 3 people out of a sample of 20 respond to the flyers they receive.

8. A family hopes to have six children. Assume boys and girls are born with the same probability.
   (a) Determine the probability that four of the children will be boys.
   (b) Determine the probability that at least two of the children will be girls.
   (c) Determine the probability that all six children will be girls.
9. **Thinking, Inquiry, Problem Solving** A study published in a consumer magazine indicated that when a husband and a wife shop for a car, the husband exerts the primary influence in the decision 70% of the time. Five couples who will be purchasing a car are selected at random. Determine the probability of each of the following.

(a) In exactly two of the couples, the husband will exert the primary influence on the decision.

(b) In all five couples, the husband will exert the primary influence on the decision.

(c) Find the expected number of couples in which the husband will exert primary influence.

(d) Determine the probability that in all five couples, the wife will exert the primary influence on the decision.

10. A baseball player has a batting average of 0.280.

(a) Find the probability that the player will get
   (i) at least 3 hits in her next 5 times at bat
   (ii) at least 3 hits in her next 10 times at bat
   (iii) at least 6 hits in her next 10 times at bat

(b) What is the player’s expected number of hits in her next 10 times at bat?

11. In a large manufacturing plant, random samples of 10 final products are taken each hour. When an hourly defect rate exceeding 3 out of 10 items is detected, production is shut down.

(a) If a production lot has a 10% defect rate, what is the probability that production will be shut down?

(b) What is the probability that production will be shut down if the actual defect rate is 30%?

(c) What is the expected number of defective items in a sample of 10 for each defect rate in parts (a) and (b)?

*Questions 12, 13, and 14 also appeared in Section 4.1. For each question, compare the simulation results you obtained in Section 4.1 with the theoretical probability that you find here.*

12. A field-goal kicker for a high school football team has an 80% success rate based on his attempts this year. Determine the probability that he will miss three field goals in a row.

13. Ten percent of the keyboards a computer company manufactures are defective. Determine the probability that one or more of the next three keyboards to come off the assembly line will be defective.

14. **Application** Imagine that the first traffic light you encounter on your way to school each morning has a 60-s cycle in which it is green for 20 s. What is the probability that you will get a green light on the next three morning trips to school?
ADDITIONAL ACHIEVEMENT CHART QUESTIONS

15. **Knowledge and Understanding** Determine the probability, correct to four decimal places, that a die rolled six times in a row will produce the following.
   (a) one 3  (b) five 3s  (c) at least two 3s

16. **Application** A multiple-choice quiz has 10 questions. Each question has four possible answers. Sam is certain that he knows the correct answer for Questions 3, 5, and 8. If he guesses on the other questions, determine the probability that he passes the quiz.

17. **Thinking, Inquiry, Problem Solving** In the dice game *Yahtzee*, a player has three tries at rolling some or all of a set of five dice. Each player is trying to achieve results such as three of a kind, two pairs, full house, so on. A yahtzee occurs when a player rolls five of a kind. If Cheryl rolls a pair of 2s on the first toss, and then rolls only the non-2s showing on the subsequent two tosses, find the probability that she gets a yahtzee.

18. **Communication** What conditions must be satisfied in order for an experiment to be considered a binomial experiment? Describe a situation that meets these conditions.

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**Chapter Problem**

Are Frog Populations Declining?

**CP3.** Suppose that the populations of the three species are distributed as shown in the following table. You capture a frog, note its species and gender, and then release it. This process is repeated until you have captured and recorded 50 frogs.

<table>
<thead>
<tr>
<th>Species</th>
<th>Percent of Total Marsh Population</th>
<th>Species Gender Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td><strong>Males</strong></td>
</tr>
<tr>
<td>Bullfrog</td>
<td>30%</td>
<td>60%</td>
</tr>
<tr>
<td>Spring peeper</td>
<td>50%</td>
<td>50%</td>
</tr>
<tr>
<td>Mink frog</td>
<td>20%</td>
<td>50%</td>
</tr>
</tbody>
</table>

(a) Determine the probability that there will be at least five female bullfrogs in the sample.
(b) Determine the probability that there will not be any mink frogs in the sample.
(c) Suppose that there were 30 spring peepers in the sample. Determine whether this is unusual enough to cause you to reconsider your original estimate of their proportion of the frog population.